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Vitali–Hahn–Saks Theorem for Hyperbolic Logics

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An analog to the Vitali–Hahn–Saks theorem for indefinite measures on hyperbolic logics of projections in indefinite metric spaces is proved.

1. INTRODUCTION

One of the problems of quantum mechanics is the description of measures on a quantum logic.

An important interpretation of a quantum logic is the set Π if all orthogonal projections of a Hilbert space H. The remarkable Gleason theorem (Gleason, 1957) says: Let H be a Hilbert space, dim $H \geq 3$, and let μ: $\Pi \to R$ be a probability measure. There exists a positive trace class operator T such that $\mu(p) = \mathrm{tr}(Tp)$, $\forall p \in \Pi$.

The problem of the construction of a quantum field theory leads to indefinite metric spaces (Dadashan and Horujy, 1983). In this case, the set P of all J-orthogonal projections serves as an analog to the logic Π. There is an indefinite analog to the Gleason theorem (Matvejchuk, 1991a,b; also see Matvejchuk, n.d.): Let H be a J-space, dim $H \geq 3$ and let μ: $P \to R$ be an indefinite measure. There exist a J-self-adjoint trace class operator T and a semitrace μ_0 such that $\mu(p) = \mathrm{tr}(Tp) + \mu_0(p)$, $\forall p \in P$.

2. A METHOD FOR CONSTRUCTING PROJECTION LOGICS

Let A be a W^*-algebra acting in a complex Hilbert space H with an inner product (., .). The sets Π and L of all orthogonal projections and of all

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projections (=idempotents) in A are important examples of orthomodular posets (= quantum logics).

Let J be a linear or conjugate linear invertible bounded operator in H. Put $[x, y] = (Jx, y)$, $\forall x, y \in H$. Let B be an algebra of bounded operators in H with the unit I closed in the weak operator topology and closed with respect to the J-adjointness, i.e., if $a \in B$, then $a^\circ \in B$, where a° is a bounded operator such that $[ax, y] = [x, a^\circ y]$, $\forall x, y \in H$. Such an algebra is called a J-algebra. Denote by P [$= P(B)$] the set of all J-self-adjoint projections from B. With respect to the ordering $p \leq g \Leftrightarrow pq = qp = p$ and the orthocomplementation $p \to p^\perp \equiv I - p$ the set P is a *quantum logic*. In general, P is not a lattice or a σ-logic.

There have been many works dedicated to J-self-adjoint operators if J is a self-adjoint (in H) operator, $J \neq \pm I$. Then there exist orthogonal projections Q^+ and Q^- such that $Q^+ + Q^- = I$, $J = Q^+ - Q^-$. Put $H^+ = Q^+H$, $H^- = Q^-H$. According to the terminology of Azizov and Iokhvidov (1989), $[., .]$ is an indefinite metric in H, J is a canonical symmetry, $H = H^+[\dot{+}]H^-$ is a canonical decomposition, and H is a Krein space (sometimes H is called a *J-space*). Let $p \in B(H)$. It is easy to see that $[pz, y] = [z, py]$, $\forall z, y \in H \Leftrightarrow p = Jp^*J$.

We shall show that there exists P which is isomorphic to Π or L.

Let $K = H \oplus H$ and let B be a W^*-algebra of operators acting in K of the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

where $a \in A$. We suppose that

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

K is a J-space with respect to the metric $[x, y] = (Jx, y)$ and B is a J-algebra in K. It is obvious that

$$P(B) = \left\{ \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, p \in \Pi \right\}$$

Hence Π and $P(B)$ are isomorphic logics.

Now let $\overline{B}$ be a W^*-algebra of operators

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad a, b \in A$$

acting in K. Let us suppose now that

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

K is a J-space with respect to the metric $[x, y] = (Jx, y)$ and $\overline{B}$ is a J-algebra in K. It is obvious that

$$P(\overline{B}) = \left\{\begin{pmatrix} q & 0 \\ 0 & q^* \end{pmatrix}, q \in L\right\}$$

Hence L and $P(\overline{B})$ are also isomorphic logics.

A specific character of J-spaces becomes fully transparent in the logic P when A is a W^*-factor and $J \in A$. In this case A is called a W^* *J-factor*. A W^* J-factor A is said to be a W^* *P-factor* if at least one of the projections Q^+ or Q^- is finite with respect to A. The logic $P(A)$ is said to be a *hyperbolic logic*.

Denote by P^+ (P^-) the set of all projections $p \in P$ for which the subspace pH is positive, i.e., ($\forall z \in pH$, $z \neq 0$, $[z, z] > 0$) (respectively, negative, i.e., $\forall z \in pH$, $z \neq 0$, $[z, z] < 0$). Any projection $e \in P$ is representable (not uniquely) in the form $e = e_+ + e_-$, where $e_+ \in P^+$, $e_- \in P^-$. Note that $p \in P^+ \Leftrightarrow pJ \geq 0$ and $p \in P^- \Leftrightarrow pJ \leq 0$.

A mapping μ: $P \to R$ is called a *measure* if $\mu(e) = \Sigma\, \mu(e_\iota)$ for any representation $e = \Sigma\, e_\iota$ (the latter sum is understood in the strong sense). A measure μ is said to be *indefinite* if $\mu/P^+ \geq 0$ and $\mu/P^- \leq 0$; *semitrace* if there exist a faithful normal semifinite trace τ on A and a number t such that $\mu(e) = \mathrm{tr}(e_+)$, $\forall e \in P$ or $\mu(e) = \mathrm{tr}(e_-)$, $\forall e \in P$. Any measure μ can be represented as a sum of the Hermitian component $\mu_h(e) = 1/2(\mu(e) + \mu(e^*))$ and the skew-Hermitian component $\mu_s = 1/2(\mu(e) - \mu(e^*))$. Clearly, if μ is an indefinite measure, then its Hermitian component is an indefinite measure, too. Note that an indefinite measure is an analog to a probability measure. Also note that a nonzero semitrace exists only on a W^* P-factor.

It was proved (Matvejchuk, 1991a,b) that for any indefinite measure μ: $P \to R$ in a W^* J-factor A (the type of A is different from I_2) there exist a J-self-adjoint linear functional $\overline{\mu} \in A_*$ and a semitrace μ_0 such that

$$\mu(e) = \overline{\mu}(e) + \mu_0(e), \qquad \forall e \in A \tag{1}$$

In addition, if μ is a Hermitian measure, then the functional $\overline{\mu}$ can be chosen so that $\overline{\mu}(\cdot) = \overline{\mu}(Q^+ \cdot Q^+) + \overline{\mu}(Q^- \cdot Q^-)$.

3. THE MAIN RESULTS

Without loss of generality we may assume that there exists a partial isometry $w \in A$ with the initial projection Q^+ and the final one $ww^* \leq Q^-$. Let $p \in P$. Then $Q^+pQ^+ = Q^+JpJQ^+ = Q^+p^*Q^+$. Hence Q^+pQ^+ and, by analogy, Q^-pQ^- are self-adjoint operators. Moreover,

$$-(Q^-pQ^+)^* = -Q^+p^*Q^- = JQ^+p^*Q^-J = Q^+Jp^*JQ^- = Q^+pQ^- \tag{2}$$

Hence

$$\begin{aligned} |Q^-pQ^+| &= ((Q^-pQ^+)^*(Q^-pQ^+))^{1/2} \\ &= (-(Q^+pQ^-)(Q^-pQ^+))^{1/2} \\ &= (Q^+p(Q^+ - I)pQ^+)^{1/2} \\ &= ((Q^+pQ^+)(Q^+pQ^+) - (Q^+pQ^+)^{1/2} \\ &= (x^2 - x)^{1/2} \end{aligned} \tag{3}$$

Here $x \equiv Q^+pQ^+$. Thus $x^2 - x \geq 0$. Denote by x_+ and x_- the positive and the negative parts of x, respectively, and by F_y the orthogonal projection onto the subspace $\overline{yH}$, $\forall y \in A$. Then we have $x_+ \geq F_{x+}$. Denote by $p \wedge Q^+$ and $p \wedge Q^-$ the orthogonal projections onto subspaces $pH \cap Q^+H$ and $pH \cap Q^-H$, respectively. Obviously $p \wedge Q^+$ ($p \wedge Q^-$) is the greatest orthogonal positive (negative) projection such that $p \wedge Q^+ \leq p$ ($p \wedge Q^- \leq p$).

Proposition 1. For any $p \in P$ the formula

$$p = x + v(x^2 - x)^{1/2} - (x^2 - x)^{1/2}v^* - v(x - F_x)v^* + p \wedge Q^- \tag{4}$$

is true. Here $x = Q^+pQ^+$, and v is a partial isometry in the polar decomposition $Q^-pQ^+ = v|Q^-pQ^+|$.

Conversely, let $x \in A$ be a self-adjoint operator such that $xQ^+ = x$; and $x_+ \geq F_{x+}$, and let v be an isometry with the initial projection $v^*v \geq F_x$ and the final one $vv^* \leq Q^-$. Then (4) defines a projection in P. In addition, if $x \geq F_x$, then $p \in P^+$ and if $x \leq 0$, then $p \in P^-$.

Proof. By (3), $Q^-pQ^+ = v(x^2 - x)^{1/2}$. By (2), $Q^+pQ^- = -(x^2 - x)^{1/2}v^*$. Now we show the equality

$$Q^-pQ^- = -v(x - F_x)v^* + p \wedge Q^-$$

For any $z \in H$ and $z_1 \in pH \cap Q^-H$ we have

$$\begin{aligned} (Q^-pQ^+ z, z_1) &= (z, (Q^-pQ^+)^*z_1) \\ = -(z, Q^+pQ^-z_1) &= -(z, Q^+z_1) = -(z, 0) = 0 \end{aligned}$$

Hence $\overline{Q^-pQ^+H} \perp p \wedge Q^-H$. In addition,

$$Q^+pQ^- = (Q^+pQ^+)(Q^+pQ^-) + (Q^+pQ^-)(Q^-pQ^-)$$

i.e.,

$$(x^2 - x)^{1/2}v^* = x(x^2 - x)^{1/2}v^* + (x^2 - x)^{1/2}v^*(Q^-pQ^-)$$

Hence

$$(x^2 - x)^{1/2}(F_x - x)v^* = (x^2 - x)^{1/2}v^*(Q^-pQ^-) \tag{5}$$

If $z \in Q^-H \ominus \overline{Q^-pH}$, then $v^*z = 0$, $p \wedge Q^-z = 0$, and $Q^-pQ^-z = 0$. This means that $Q^-pQ^-z = -v(x - F_x)v^*z + p \wedge Q^-z$.

If $z \in p \wedge Q^-H$, then $v^*z = 0$ and

$$Q^-pQ^-z = z = p \wedge Q^-z - v(x - F_x)v^*z$$

If

$$z \in \overline{Q^-pH} \ominus p \wedge Q^-H$$

then

$$v^*z \in \overline{(x^2 - x)^{1/2}H} \qquad \text{and} \qquad p \wedge Q^-z = 0$$

The equality (5) implies that

$$v^*Q^-pQ^-z = (F_x - x)v^*z = (F_x - x)v^*z + p \wedge Q^-z$$

i.e., $-v(x - F_x)v^*z + p \wedge Q^-z = Q^-pQ^-z$, $\forall z \in H^-$.

Now let $x \in A$ be a self-adjoint operator such that $xQ^+ = x$, $x_+ \geq F_{x+}$, and let $v \in A$ be an isometry with the initial projection $v^*v \geq F_x$ and the final one $vv^* \leq Q^-$. Using the right side of (4), define an operator p. It can be easily verified that $p^2 = p$ and $Jp^*J = p$. Hence $p \in P$. If $x \geq F_x$, then $p^*Jp \geq 0$. Hence

$$[pz, pz] = (Jpz, pz) = (p^*Jpz, z) \geq 0, \qquad \forall z \in H$$

Thus $p \in P^+$. If $x \leq 0$, then $p^*Jp \leq 0$. Hence $p \in P^-$. QED

In the sequel a projection p of the form

$$p = x + v(x^2 - x)^{1/2} - (x^2 - x)^{1/2}v^* - v(x - F_x)v^*$$

will be denoted by $p(x, v)$. Let x and v be from Proposition 1. It is clear that $p = p(x_+, v) + p(x_-, v) + p \wedge Q^-$. It is directly verified that $\|p(x, v)\| = \|2x - I\|$. Thus the following remark is true.

Remark 1. $\|p(x, v)\| = \max\{\|p(x_+, v)\|, \|p(x_-, v)\|\}$.

Theorem 1. Let P be the set of all J-self-adjoint projections from a semifinite W^* J-factor and let μ^n: $P \to R$ be a sequence of indefinite measures. Assume that for every projection $p \in P$ there exists a finite

$$\lim_{n\to\infty} \mu^n(p) \equiv \mu(p)$$

Then $\tau(p_\alpha) \to_\alpha 0$ implies $\|p_\alpha\|^{-1}\mu^n(p_\alpha) \to_\alpha 0$ uniformly in n. In addition, if A is a finite factor, then μ: $P \to R$ is an indefinite measure.

Proof. The proof will consist of two steps.

1. We first prove an analog to this theorem for a measures on Π. Let ν^n: $\Pi \to R$ be a sequence of measures such that $\forall p \in \Pi$ there exists a finite $\lim_{n\to\infty}\nu^n(p) = \nu(p)$. By the analog to the Gleason theorem [see a sketch of the proof in Matvejchuk (1988) and the full proof in Matvejchuk (1987)], there exists a sequence of self-adjoint ultraweakly continuous linear functionals $\overline{\nu^n}$ such that $\overline{\nu^n}(p) = \nu(p)$, $\forall p \in \Pi$. Next, by contradiction and Lemma 5.5 (Takesaki, 1979, Chapter 3), we can obtain the assertion for measures on Π.

Still, we would like to give another proof which almost completely repeats the classical one (Dunford and Schwartz, 1958, Chapter 3, Section 7). We do not here need the aforementioned analog to the Gleason theorem. We make use of the continuity of ν^n in the norm $\|a\|_1 = \tau(|a|)$, $a \in A$ alone. Denote by M_1 the set $\{e \in \Pi: \tau(e) < +\infty\}$. Let $\epsilon > 0$ be arbitrary. The set

$$M_k(\epsilon) = \{p \in M_1: \sup_{n\geq 1}|\nu^k(p) - \nu^{k+n}(p)| \leq \epsilon\}$$

is closed in the norm $\|\cdot\|_1$. By construction, $M_1 = \cup_{k\geq 1} M_k(\epsilon)$. The set M_1 is a complete metric space with respect to the norm $\|\cdot\|_1$. Thus M_1 is a set of the second category. Hence at least one of the sets $M_k(\epsilon)$, say, $M_{k_0}(\epsilon)$, contains a nonempty open set M ($\subseteq M_1$). Hence there exist $p_0 \in M_1$ and $\delta > 0$ such that $\|p - p_0\|_1 \leq \delta$ implies $\sup_{n\geq 1}|\nu^{k_0}(p) - \nu^{k_0+n}(p)| \leq \epsilon$.

Let now $\tau(g) \leq \delta$, and $g \in M_1$. Denote by g' the orthogonal projection onto $\overline{p_0 g p_0 H}$. Then $\tau(g') \leq \delta$. Hence for $p_1 \equiv p_0 - g'$ and $p_2 \equiv p_1 + g'$ we have $\|p_1 - p_0\|_1 \leq \delta$ and $\|p_2 - p_0\|_1 \leq \delta$. In addition, $p_2 - p_1 = g$. Thus for $k \geq k_0$ the following inequality holds:

$$\begin{aligned} |\nu^k(g)| &\leq |\nu^{k_0}(g)| + |\nu^{k_0}(g) - \nu^k(g)| \\ &\leq |\nu^{k_0}(g)| + |\nu^{k_0}(p_2) - \nu^k(p_2)| + |\nu^{k_0}(p_1) - \nu^k(p_1)| \\ &\leq |\nu^{k_0}(g)| + 2\epsilon \end{aligned} \tag{6}$$

Observe that $\tau(e_\alpha) \to_\alpha 0$ ($e_\alpha \in \Pi$) implies $\nu^{k_0}(e_\alpha) \to_\alpha 0$. Hence by (6), $\tau(e_\alpha) \to_\alpha 0$ implies $\nu^k(e_\alpha) \to_\alpha 0$ uniformly in k. It is obvious that the function ν: $\Pi \to R$ is finitely additive. If $\tau(I) < \infty$, then ν is completely additive by the above. Thus ν is a measure.

2. Now, let μ^n: $P \to R$ be a sequence of indefinite measures from the assumption of the theorem. It is clear that the sequence (μ_h^n) of the Hermitian components of (μ^n) satisfies the assumption also. By (1),

$$\overline{\mu_h^n}(e) = \mu_h^n(e) = \mu^n(e) \leq 0, \qquad \forall e \in P, \quad e \leq Q^-$$

By the assumption, there exists a finite

$$\lim \overline{\mu_h^n}(e), = \lim \overline{\mu_h^n}\,(Q^- e Q^-), \qquad \forall e \in P, \quad e \leq Q^-$$

Hence there exists a finite $\lim \overline{\mu_h^n}(a)$, $\forall a \in A$, $a = Q^- a Q^-$.

Let a net $(p^\alpha) \subset P$ be such that $\tau(p^\alpha) \to 0$. Denote by e_α the orthogonal projection onto $\overline{Q^- p^\alpha H}$. Then $-\|p^\alpha\| e_\alpha \leq Q^- p^\alpha Q^- \leq \|p^\alpha\| e_\alpha$. Hence

$$|\overline{\mu_h^n}(Q^- p^\alpha Q^-)| \leq \|p^\alpha\| |\overline{\mu_h^n}(e_\alpha)|$$

By this inequality and step 1, $\tau(p^\alpha) \to_\alpha 0$ implies

$$\|p^\alpha\|^{-1} |\overline{\mu_h^n}(Q^- p^\alpha Q^-)| \to_\alpha 0 \qquad \text{uniformly in } n$$

Let $e \in P$, $e \leq Q^+$, and $x_0 = \beta e$, $\beta < 0$. Also let v_0 be a partial isometry with the initial projection e and the final one $\leq Q^-$, and let $p = p(x_0, v_0)$. By Proposition 1, $p \in P^-$. There exists a finite

$$\begin{aligned} &\lim_{n\to\infty}(\mu_h^n(p) - \overline{\mu_h^n}(Q^- p Q^-)) \\ &= \lim(\overline{\mu_h^n}(p) + \mu_0^n(p) - \overline{\mu_h^n}(Q^- p Q^-)) \\ &= \lim \overline{\mu_h^n}(Q^+ p Q^+) \\ &= \beta \lim \overline{\mu_h^n}(e), \qquad \forall e \leq Q^+ \end{aligned}$$

Hence there exists a finite $\lim \mu_0^n(e) = \lim t_n \tau(e)$. Again, by analogy with $\overline{\mu_h^n}(Q^- \cdot Q^-)$, we obtain that $\tau(p^\alpha) \to_\alpha 0$ implies $\|p^\alpha\|^{-1} |\overline{\mu_h^n}(Q^+ p^\alpha Q^+)| \to_\alpha 0$, $\|p^\alpha\|^{-1} |\mu_0^n(p^\alpha)| \to_\alpha 0$, and thus $\|p^\alpha\|^{-1} |\mu_h^n(p^\alpha)| \to_\alpha 0$ uniformly in n.

Consider now the sequence (μ_s^n) of the skew-Hermitian components of (μ^n). For any $p \in P^-$ we have $p^* \in P^-$,

$$0 \geq \mu(p) = \mu_h(p) + \mu_s(p)$$

and

$$0 \geq \mu(p^*) = \mu_h(p^*) + \mu_s(p^*) = \mu_h(p) - \mu_s(p)$$

Hence

$$|\mu_h(p)| = -\mu_h(p) \geq |\mu_s(p)| \tag{7}$$

By analogy,

$$|\mu_s(p)| \leq \mu_h(p), \qquad \forall p \in P^+ \tag{8}$$

Let $p^\alpha = p(x^\alpha, v^\alpha) + p^\alpha \wedge Q^-$. Put $p_+^\alpha = p(x_+^\alpha, v^\alpha)$ and $p_-^\alpha = p(x_-^\alpha, v^\alpha) + p^\alpha \wedge Q^-$. By (7), (8), and Remark 1, we have

$$\begin{aligned} &\|p^\alpha\|^{-1} |\mu_s^n(p^\alpha)| \\ &\leq \|p^\alpha\|^{-1}(|\mu_s^n(p_+^\alpha)| + |\mu_s^n(p_-^\alpha)|) \\ &\leq \|p_+^\alpha\|^{-1} |\mu_s^n(p_+^\alpha)| + \|p_-^\alpha\|^{-1} |\mu_s^n(p_-^\alpha)| \\ &\leq \|p_+^\alpha\|^{-1} |\mu_h^n(p_+^\alpha)| + \|p_-^\alpha\|^{-1} |\mu_h^n(p_-^\alpha)| \end{aligned}$$

Hence if $\tau(p^\alpha) \to_\alpha 0$, then $\|p^\alpha\|^{-1}|\mu_s^n(p^\alpha)| \to_\alpha 0$ uniformly in n. Finally,

$$\|p^\alpha\|^{-1}\mu(p^\alpha) = \|p^\alpha\|^{-1}(\mu_h^n(p^\alpha) + \mu_s^n(p^\alpha)) \to_\alpha 0 \qquad \text{uniformly in } n$$

Now assume that $\tau(I) < \infty$. Put $\mu(p) = \lim \mu^n(p)$, $\forall p \in P$. It is clear that $\mu/P^+ \geq 0$ and $\mu/P^- \leq 0$. Let $e = \Sigma_{i=1}^{\infty} e_i$, where $e, e_i \in P$. Then $p_k = \Sigma_k^{\infty} e_i \to 0$ in the strong sense and $\tau(p_k) \to 0$. There exists $r \geq 1$ such that $1 \leq \|p_k\| \leq r$, $\forall k$. By the above, $\mu^n(p_k) \to_k 0$ uniformly in n. Hence $\mu(p_k) \to_k 0$. This means $\mu(e) = \Sigma\, \mu(e_i)$. Therefore, μ: $P \to R$ is an indefinite measure. QED

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